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A BERNOULLI TYPE EXPANSION OF THE INVERSE TAYLOR OPERATOR, AND SOME OF ITS APPLICATIONS

DEMETRIOS P. KANOUSSIS

American University of Athens (AUA EUROPE)
Athens
Greece
e-mail: dkanoussis@teemail.gr

Abstract

A Bernoulli type expansion of the inverse Taylor operator is introduced, and its application in solving the difference equation $\Delta y(x) = g(x)$, is investigated. Further applications relate to the Bernoulli and Euler polynomials, and to the evaluation of sums of powers of positive integers.

1. Introduction

While the published research on obtaining solution (exact and/or approximate) to various types of differential equations is quite extensive, the corresponding research for difference equations, is rather limited.

The difference equation

$$\Delta y(x) = y(x+1) - y(x) = g(x), \tag{1.1}$$

where $g(x)$ is given, was studied by Krull, in his pioneer work [7], and later on by other researchers [4], [5], [8], and [9].

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In a recent paper [6], a method for obtaining the solution of the difference equation (1.1), was developed, for a broad class of functions $g(x)$. This method is based on the operator equivalency (Taylor formula),

$$\Delta = e^{D} - I \text{ or } D = \ell n(I + \Delta), \tag{1.2}$$

where Δ is the difference operator introduced in (1.1), $D = d / dx$ is the derivative operator, and I is the identity operator.

Assuming that $G(x)$ is an antiderivative of $g(x)$, and in view of (1.2), the difference equation (1.1) is formally equivalent to

$$\Delta y(x) = \ell n(I + \Delta)G(x). \tag{1.3}$$

This approach was exploited in [6] and proved to be fruitful, since among other things, it revealed some interesting expansions for the gamma and the digamma (or psi) functions.

In this paper, we investigate an alternative approach of treating the Equation (1.1). Making use of (1.2), we write (1.1) as

$$\left(e^{D} - I\right)y(x) = g(x) = DG(x), \tag{1.4}$$

or solving formally for $y(x)$,

$$y(x) = \left(e^{D} - I\right)^{-1} DG(x). \tag{1.5}$$

Since

$$\frac{z}{e^{z} - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k = 1 + \sum_{k=1}^{\infty} \frac{B_k}{k!} z^k, \quad |z| < 2\pi, \tag{1.6}$$

where B_k is the k-th Bernoulli number, one can expand formally $\left(e^{D} - I\right)^{-1} D,$ as

$$\left(e^{D} - I\right)^{-1} D = I + \sum_{k=1}^{\infty} \frac{B_k}{k!} D^k, \tag{1.7}$$

and applying (1.7) into (1.5), yields

$$y(x) = G(x) + \sum_{k=1}^{\infty} \frac{B_k}{k!} D^k G(x) = G(x) + \sum_{k=1}^{\infty} \frac{B_k}{k!} D^{k-1} g(x). \tag{1.8}$$

Formula (1.8) suggests a method of solving the difference equation (1.1).

In the sequel of this note, we will focus our attention in cases where $g(x)$ is a polynomial of x, and investigate some rather interesting applications, which naturally follow.

2. The Case where $g(x)$ is a Polynomial

If $g(x)$ is a polynomial of degree n, then the infinite series in (1.8) terminates (i.e., becomes a finite sum), the last term being

$$\frac{B_{n+1}}{(n+1)!} D^{n+1} G(x) = \frac{B_{n+1}}{(n+1)!} D^n g(x) = \frac{B_{n+1}}{(n+1)!} g^{(n)}(x).$$

We are thus led to the following:

Theorem 1. *If $g(x)$ is a polynomial of degree n and $G(x)$ is an antiderivative of $g(x)$, then the solution of the difference equation $\Delta y(x) = g(x)$, can be expressed as*

$$y(x) = G(x) + \sum_{k=1}^{n+1} \frac{B_k}{k!} g^{(k-1)}(x). \tag{2.1}$$

Having obtained (2.1) *as the solution to the difference equation $\Delta y(x) = g(x)$, one may easily generalize to the difference equation*

$$y(x+\lambda) - y(x) = g(x), \text{ where } \lambda = 1, 2, 3, \ldots. \tag{2.2}$$

Theorem 2. *If $g(x)$ is a polynomial of degree n and $G(x)$ is an antiderivative of $g(x)$, then the solution of the difference equation $y(x+\lambda) - y(x) = g(x)$, $\lambda = 1, 2, 3, \ldots$, can be expressed as*

$$y(x) = \frac{G(x)}{\lambda} + \sum_{k=1}^{n+1} \lambda^{k-1} \frac{B_k}{k!} g^{(k-1)}(x). \tag{2.3}$$

Proof. If we set $x = \lambda \cdot w$, Equation (2.2) becomes

$y(\lambda(w+1)) - y(\lambda w) = g(\lambda w)$, and if we call

$\widetilde{y}(w) = y(\lambda w)$ and $\widetilde{g}(w) = g(\lambda w)$, the last equation becomes

$\widetilde{y}(w+1) - \widetilde{y}(w) = \widetilde{g}(w)$, having solution, according to (2.1),

$$\widetilde{y}(w) = \widetilde{G}(w) + \sum_{k=1}^{n+1} \frac{B_k}{k!} \widetilde{g}^{(k-1)}(w), \tag{2.4}$$

where $\widetilde{G}(w)$ is an antiderivative of $\widetilde{g}(w)$, or since $w = \frac{x}{\lambda}$,

$y(x) = \frac{G(x)}{\lambda} + \sum_{k=1}^{n+1} \lambda^{k-1} \frac{B_k}{k!} g^{(k-1)}(x)$, and this completes the proof.

We are now in a position to state two fundamental theorems, applications of which will be given in the next section.

Theorem 3. *Given a polynomial* $q(x)$, *of degree* n, *the polynomial* $E(x)$, *of degree* n, *which satisfies the equation*

$$E(x) + E(x+1) + E(x+2) + \ldots + E(x+\lambda) = q(x), \tag{2.5}$$

λ *being any positive integer*, *can be expressed as*

$$E(x) = \frac{\Delta Q(x)}{\lambda + 1} + \sum_{k=1}^{n} (\lambda+1)^{k-1} \frac{B_k}{k!} \Delta\left(q^{(k-1)}(x)\right), \tag{2.6}$$

where, *of course*, $Q(x)$ *is an antiderivative of* $q(x)$.

Proof. From Equation (2.5), the polynomial $E(x)$ satisfies the difference equation

$$E(x+\lambda+1) - E(x) = q(x+1) - q(x) = \Delta q(x),$$

where $\Delta q(x)$ is a polynomial of degree $n-1$. Making use of (2.3), and the fact that $(\Delta q(x))^{(k-1)} = \Delta\left(q^{(k-1)}(x)\right)$, $k = 1, 2, 3, \ldots,$ Equation (2.6) is readily obtained, and the proof is completed.

It will be shown shortly, that this approach furnishes a simple and natural way to develop the theory of Euler polynomials.

Theorem 4. *Given a polynomial* $g(x)$, *of degree* n, *and a positive integer* λ, *then*

$$\sum_{i=0}^{\lambda} g(x+i) = G(x+\lambda+1) - G(x) + \sum_{k=1}^{n} \frac{B_k}{k!}\{g(x+\lambda+1) - g(x)\}^{(k-1)}, \quad (2.7)$$

where $G(x)$ *is an antiderivative of* $g(x)$.

Proof. Let $y(x) = \sum_{i=0}^{\lambda} g(x+i) = g(x) + g(x+1) + \ldots + g(x+\lambda)$. Then $\Delta y(x) = y(x+1) - y(x) = g(x+\lambda+1) - g(x)$. The expression $g(x+\lambda+1) - g(x)$ is a polynomial of degree $n-1$, so according to (2.1), $y(x)$ can be expressed as

$$y(x) = \sum_{i=0}^{\lambda} g(x+i) = G(x+\lambda+1) - G(x) + \sum_{k=1}^{n} \frac{B_k}{k!}\{g(x+\lambda+1) - g(x)\}^{(k-1)},$$

and the proof is completed.

3. Evaluation of the Sum $1^n + 2^n + \ldots + \lambda^n$, n being any Positive Integer

While the problem of evaluating the sum

$$S_n(\lambda) = \sum_{j=1}^{\lambda} j^n = 1^n + 2^n + 3^n + \ldots + \lambda^n, \quad (3.1)$$

n being a positive integer, is quite old, still there is an interest in the problem and various papers are being published on the subject [2], [3], and [10].

In this paper, we present a new, as far as we know, method to obtain an expression for $S_n(\lambda)$.

Let

$$\Phi(x) = \sum_{j=0}^{\lambda} (x+j)^n = x^n + (x+1)^n + (x+2)^n + \ldots + (x+\lambda)^n. \tag{3.2}$$

From Theorem 4, Equation (2.7), $\Phi(x)$ can be expressed equivalently as

$$\Phi(x) = \frac{(x+\lambda+1)^{n+1} - x^{n+1}}{n+1} + \sum_{k=1}^{n} \frac{B_k}{k!}\left((x+\lambda+1)^n - x^n\right)^{(k-1)}. \tag{3.3}$$

The term $\frac{B_k}{k!}\left((x+\lambda+1)^n - x^n\right)^{(k-1)}$, can be simplified as follows:

$$\frac{B_k}{k!} \cdot n(n-1)(n-2)\ldots(n-(k-2))\left((x+\lambda+1)^{n+1-k} - x^{n+1-k}\right)$$

$$= B_k \frac{1}{n+1}\binom{n+1}{k}\left((x+\lambda+1)^{n+1-k} - x^{n+1-k}\right).$$

Therefore, Equation (3.3) can be written, equivalently,

$$\Phi(x) = \frac{1}{n+1}\left\{(x+\lambda+1)^{n+1} - x^{n+1} + \sum_{k=1}^{n} B_k \binom{n+1}{k}\left((x+\lambda+1)^{n+1-k} - x^{n+1-k}\right)\right\},$$

or, in a more compact form

$$\Phi(x) = \frac{1}{n+1}\sum_{k=0}^{n} B_k \binom{n+1}{k}\left((x+\lambda+1)^{n+1-k} - x^{n+1-k}\right), \tag{3.4}$$

where $B_0 = 1$.

Upon noticing that $\Phi(0) = S_n(\lambda) = 1^n + 2^n + \ldots + \lambda^n$, one easily obtains that

$$S_n(\lambda) = \frac{1}{n+1}\sum_{k=0}^{n} B_k \binom{n+1}{k}(\lambda+1)^{n+1-k}. \tag{3.5}$$

It is worth mentioning that (3.4) could be used in order to evaluate other arithmetic sums, such as $1^n + 3^n + 5^n + \ldots + (2\lambda + 1)^n$, by evaluating $\Phi(x)$ at $x = \frac{1}{2}$, or $1^n + 4^n + 7^n + \ldots + (3\lambda + 1)^n$, by evaluating $\Phi(x)$ at $x = \frac{1}{3}$, or in general, $1^n + (1 + 1 \cdot m)^n + (1 + 2 \cdot m)^n + \ldots + (1 + \lambda \cdot m)^n$, m being a positive integer ≥ 2, by evaluating $\Phi(x)$ at $x = \frac{1}{m}$.

It is a trivial matter to show, for example, that

$$\sum_{k=0}^{\lambda}(1 + k \cdot m)^n = \frac{1}{n+1}\sum_{k=0}^{n} B_k \binom{n+1}{k}\left((1 + (\lambda + 1)m)^{n+1-k} - 1\right) \cdot m^{k-1}. \tag{3.6}$$

4. The Bernoulli and Euler Polynomials

The theory developed thus far, furnishes a particularly simple and natural way to investigate polynomials defined by means of a recursive formula.

It is a well-known fact, that the Bernoulli and Euler polynomials can be introduced in various ways.

In this paper, we will define the Bernoulli and the Euler polynomials, by means of their respective recursive formulas, apply the theory developed, and obtain quite easily their corresponding expansions.

The Bernoulli polynomial of degree n, can be defined as follows:

$$B_n(x + 1) - B_n(x) = n \cdot x^{n-1}, \quad B_n(0) = B_n, \quad n = 0, 1, 2, \ldots, \tag{4.1}$$

where B_n is the n-th Bernoulli number.

Making use of (2.1), it is easily verified that

$$B_n(x) = c + \sum_{k=0}^{n} B_k \binom{n}{k} x^{n-k}, \quad B_0 = 1, \tag{4.2}$$

where c is the arbitrary constant of integration. Since $B_n(0) = B_n$, it turns out that $c + B_n\binom{n}{n} = B_n$, i.e., $c = 0$, and finally,

$$B_n(x) = \sum_{k=0}^{n} B_n \binom{n}{k} x^{n-k}, \quad n = 0, 1, 2, \dots. \tag{4.3}$$

On the other hand, the Euler polynomial of degree n can be defined by means of the formula [1]

$$E_n(x+1) + E_n(x) = 2 \cdot x^n, \quad n = 0, 1, 2, \dots. \tag{4.4}$$

The polynomial $E_n(x)$, for $n \geq 1$, can now be easily calculated, from (2.6), with $\lambda = 1$ and $q(x) = 2 \cdot x^n$. [The $E_0(x) = 1$, as it is easily verified from (4.4)].

Straightforward application of (2.6) yields

$$E_n(x) = \frac{(x+1)^{n+1} - x^{n+1}}{n+1} + \sum_{k=1}^{n} 2^{k-1} \frac{B_k}{k!} \left((x+1)^n - x^n\right)^{(k-1)},$$

on equivalently, after some simplifications

$$E_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} 2^k B_k \binom{n+1}{k} \left((x+1)^{n+1-k} - x^{n+1-k}\right). \tag{4.5}$$

For example, for $n = 0, 1, 2,$ and 3, we obtain the corresponding Euler polynomials

$$E_0(x) = 1,\ E_1(x) = x - \frac{1}{2},\ E_2(x) = x^2 - x,\ E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \text{ etc.}$$

Various properties of Euler's polynomials can be derived, rather easily, from (4.5). For example, one can show that if n is an odd positive integer, i.e., $n = 1, 3, 5, \dots$, then

$$E_n\left(-\frac{1}{2}\right) = -\frac{1}{2^{n-1}}. \tag{4.6}$$

Indeed, applying (4.5) for $x = -\frac{1}{2}$, and assuming n to be an odd positive integer, then the only surviving term in (4.5) will be the one corresponding to $k = 1$, (since $B_3 = B_5 = B_7 = \ldots = 0$, while for the terms $B_2, B_4, B_6, \ldots$, which are not zero, the term $\left((x+1)^{n+1-k} - x^{n+1-k}\right)$ evaluated at $x = -\frac{1}{2}$, vanishes). The first term $\frac{1}{n+1}\left((x+1)^{n+1} - x^{n+1}\right)$, also vanishes, when evaluated at $x = -\frac{1}{2}$.

Therefore, $E_n\left(-\frac{1}{2}\right) = \frac{1}{n+1} \cdot 2 \cdot B_1 \cdot \binom{n+1}{1}\left(\left(\frac{1}{2}\right)^n - \left(-\frac{1}{2}\right)^n\right) = -\frac{1}{2^{n-1}}$, on the assumption of course, that n is an odd positive integer, $\left(B_1 = -\frac{1}{2}\right)$.

Now making use of the defining Equation (4.4) and applying at $x = -\frac{1}{2}$, one obtains

$$E_n\left(\frac{1}{2}\right) + E_n\left(-\frac{1}{2}\right) = 2 \cdot \left(-\frac{1}{2}\right)^n. \tag{4.7}$$

If n is odd positive, then from (4.6) and (4.7), one obtains that $E_n\left(\frac{1}{2}\right) = 0$, $n = 1, 3, 5, \ldots$.

The Euler numbers are defined as [1]

$$E_n = 2^n \cdot E_n\left(\frac{1}{2}\right). \tag{4.8}$$

If n is odd positive, then $E_n = 0$, while if n is an even positive integer, the following expression can be found, using (4.5) and (4.8)

$$E_n = \frac{1}{n+1} \sum_{k=0}^{n} 2^{2k-1} B_k \binom{n+1}{k}\left(3^{n+1-k} - 1\right), \tag{4.9}$$

for $n = 0, 2, 4, 6, \ldots$.

We have thus expressed the Euler numbers, explicitly, in terms of the Bernoulli numbers.

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